Low Complexity Homeomorphic Projection to Ensure Neural-Network Solution Feasibility for Optimization over (Non-)Convex Set

ENMING LIANG, City University of Hong Kong, China
MINGHUA CHEN, City University of Hong Kong, China
STEVEN H. LOW, California Institute of Technology, USA

There has been growing interest in employing neural networks (NN) to directly solve constrained optimization problems with low run-time complexity. However, it is non-trivial to ensure NN solutions strictly satisfy problem constraints due to inherent NN prediction errors. Existing feasibility-ensuring methods are either computationally expensive or lack performance guarantee. In this paper, we propose *homeomorphic projection* [3] as a low-complexity scheme to guarantee NN solution feasibility for optimization over a general set homeomorphic to a unit ball, potentially nonconvex. The idea is to (i) learn a minimum distortion homeomorphic mapping between the constraint set and a unit ball by invertible NN (INN), and then (ii) perform a simple bisection operation concerning the unit ball so that the INN-mapped final solution is feasible with respect to the constraint set with minor distortion-induced optimality loss. We prove the feasibility guarantee and bound the optimality loss. Simulation results show that homeomorphic projection outperforms existing methods in feasibility and run-time complexity, while achieving similar optimality loss.

CCS Concepts: • **Theory of computation → Continuous optimization**; • **Computing methodologies → Neural networks**.

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1 INTRODUCTION

Constrained Optimization (CO) has tremendous applications in various engineering domains, including supply chain, transportation, power system, and system resource allocation. A large number of iterative algorithms have been developed and incorporated into commercial solvers (e.g., Gurobi) to solve various CO problems. While widely successful, iterative algorithms can still fail to solve challenging CO problems in real-time, limiting their usefulness in time-sensitive applications. Recently, NN-based schemes have been developed for solving CO in real-time, including the end-to-end (E2E) solution mapping [1] and the learning-to-optimize (L2O) iterative scheme [2]. However, it is non-trivial to ensure NN solution feasibility with respect to the problem constraints, due to inherent prediction errors. Existing feasibility-ensuring methods, e.g., penalty, sampling, and projection approaches, are either computationally expensive or lack performance guarantee.

Authors' addresses: Enming Liang, City University of Hong Kong, Hong Kong, China, eliang4-c@my.cityu.edu.hk; Minghua Chen, City University of Hong Kong, Hong Kong, China, minghua.chen@cityu.edu.hk; Steven H. Low, California Institute of Technology, California, USA, slow@caltech.edu.

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In this paper, we develop *homeomorphic projection* (HP) as the first work to guarantee NN solution feasibility for (fairly) general constrained optimization problems, with bounded optimality loss and low run-time complexity.

2 HOMEOMORPHIC PROJECTION

We consider the continuous constrained optimization problem: $\min_{x\in\mathbb{R}^n} \; f(x,\theta), \;$ s.t. $\; x \in \mathcal{K}_\theta$, where θ is the input parameters and x is the decision variables. We assume the constrained set is homeomorphic to a unit ball as $\mathcal{K}_\theta = \psi_\theta(\mathcal{B})$, where $\psi_\theta \in \mathcal{H}^n$ is a homeomorphic mapping (continuous bijection with continuous inverse). This assumption covers (i) any compact and convex set and (ii) a class of compact and simply-connected non-convex sets (e.g., star set).

We propose to learn the minimum-distortion-homeomorphic (MDH) mapping between them by the following problem: $\min_{\psi_\theta\in\mathcal{H}^n} \; \log \mathrm{D}(\psi_\theta^{-1}, \mathcal{X}_\theta), \;$ s.t. $\; \mathcal{K}_\theta = \psi_\theta(\mathcal{B})$, where $\mathrm{D}(\psi_\theta^{-1}, \mathcal{X}_\theta)$ indicates the distortion of the mapping ψ_θ^{-1} over support set $\mathcal{X}_\theta$. We then leverage invertible neural network (INN) Φ_θ to approximate the MDH mapping in an unsupervised manner.

After training, given an infeasible solution $\tilde{x}_\theta$, we proposed a bisection algorithm to recover its feasibility as $\hat{x}_\theta = \Phi_\theta(\alpha^*\tilde{z}_\theta) \in \mathcal{K}_\theta$, where $\tilde{z}_\theta = \Phi_\theta^{-1}(\tilde{x}_\theta)$ and $\alpha^* = \sup_{\alpha\in[0,1]}\{\Phi_\theta(\alpha\tilde{z}_\theta) \in \mathcal{K}_\theta\}$ is solved efficiently through bisection.

3 MAIN RESULTS

THEOREM 3.1. *Given an infeasible* $\tilde{x}_\theta$ *with bounded prediction error* $\epsilon_{\text{pre}} = \sup_{\theta\in\Theta} \|\tilde{x}_\theta - x_\theta^*\|$, *and a valid m-layer INN mapping* $\Phi_\theta \in \mathcal{H}^n$ *with approximation error* ϵ_{inn} *and distortion* $\mathrm{D}(\Phi_\theta^{-1}, \mathcal{Y}_\theta)$, *the bisection algorithm with maximum k steps will return a solution* $\hat{x}_\theta^k$ *such that:*

- *it is guaranteed to be feasible, i.e.,* $\hat{x}_\theta^k \in \mathcal{K}_\theta$*;*
- *it has a bounded optimality loss as* $\|\hat{x}_\theta^k - x_\theta^*\| \leq \epsilon_{\text{pre}} + \mathrm{D}(\Phi_\theta^{-1}, \mathcal{Y}_\theta)(2\epsilon_{\text{inn}} + 3\epsilon_{\text{pre}} + \delta_{\text{bis}}^k)$, *where* $\delta_{\text{bis}}^k = 2^{-k}(\mathrm{diam}(\mathcal{K}_\theta) + \epsilon_{\text{pre}})$ *and* $\mathcal{Y}_\theta = \mathcal{K}_\theta + \mathcal{B}(0, \max\{\epsilon_{\text{pre}}, \epsilon_{\text{inn}}\})$*;*
- *it has a run-time complexity as* $\mathcal{O}(k(mn^2 + Gn_{\text{ineq}}))$*.*

We test the HP framework over SDP and AC-OPF problems. The results (feasibility rate, solution MAPE, objective MAPE, and speedup) are shown in the following table:

Table 1. Performance in constrained optimization problems

Method	Feasibility %	Solution Δ%	Objective Δ%	Speedup ×	Feasibility %	Solution Δ%	Objective Δ%	Speedup ×
	SDP: $n = 15 \times 15$, $n_{\text{eq}} = 100$, $n_{\text{ineq}} = 1$				**AC-OPF**: $n = 344$, $n_{\text{eq}} = 236$, $n_{\text{ineq}} = 452$			
NN	45.02	5.92	1	31202.6	73.24	1.27	0.24	178.2
NN+WS	100	2.62	0.4	0.9	100	0.94	0.18	3.6
NN+Proj	100	5.12	1.72	1.2	100	1.55	0.24	3.8
NN+D-Proj	68.65	5.9	0.99	3.6	87.79	1.26	0.24	4.9
NN+H-Proj	100	6.51	1.19	149.2	100	1.58	0.51	24.6

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