Online Conversion under Unknown Horizon: Beating $1 + \ln \theta$ with a Single Trusted Query

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It has been known that a few bits of queries of the maximum price can break the lower bound of online conversion problem. We ask if a query only asks about the relationship of the future price and the current price plus a constant (k), will it also break the lower bound in the unknown horizon case? It turns out that when $k = 0$, the query model is not helpful. We present a deterministic algorithm and a randomized algorithm to effectively utilize the prediction and break the lower bound when $k > 0$.

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1 INTRODUCTION

Imagine that you play a financial game and own one unit of asset at the outset. Prices of the asset change over time, and only a price range $[p_{\min}, p_{\max}]$ is known. You must seize the opportunity to trade each time a new price appears because the game could end anytime. If you are reminded when the game ends and get to trade the remaining asset at the last price shown, this game is known as online conversion problems [5, 6]. In this work, we consider the online conversion problem under unknown horizon where the game ends quietly, possibly leading to assets not traded.

The above example of online conversion problem is one example of decision making under the uncertainty of future, which has been of dramatic interest to both the academia and the industry. Among others, online algorithm and competitive analysis is one classical theoretical framework that characterizes the performance loss of any algorithm that makes decision based on only causal information compared to a performance optimizer equipped with full knowledge of future [3]. The metric typically used is the competitive ratio, defined as $\max_I \frac{\mathsf{OPT}(I)}{\mathsf{ALG}(I)}$. It is obvious that a competitive ratio closer to 1 is better. For the online conversion problem under unknown horizon, it has been shown that the best possible competitive ratio any online algorithm can reach is $1 + \ln \theta$ [4], where $\theta := p_{\max}/p_{\min}$ denotes the fluctuation ratio of the prices.

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Now imagine that the financial game grants you with one opportunity of asking a question with a binary answer. The answer is guaranteed to be always correct. What can be the question you ask to improve the competitive ratio? We formally provide one set of answers to it in this work. The setting described here is known as query-based models for online problems, which aims to assist online algorithms with the chance of querying an oracle, and hopefully break the barrier of existing lower bounds. In contrast to the feverish trend in the other genre of online algorithms with error-prone machine-learned predictions, we are not aware of any work on query-based online conversion problems except for [1], which studied the impact of using multiple error-prone binary queries on the maximum price in the future. In comparison, we are curious to see how **one** bit of trusted query can improve the competitive ratio.

The first question is, what should this one bit describe? In financial markets, sometimes finding the information about the maximum upcoming exchange rate is not easy. Some local vision with the centrality of the current price is probably more accessible instead. More precisely, assume a comparison of the maximum price with the currently seen price is given to us. This motivates us to come up with the following query-based model dependent on the prices seen. We call it **instance-dependent** query model. In each round, you have the chance to use your one and only query and ask the oracle this question: *Is the unknown maximum price more than the current price plus some constant?* This constant serves as a measure of the advantage of trading later over trading now. When the constant (call it k) is zero, this model coincides with the following natural question: "*is there a higher price in the future*?" When the answer is no, we should convert all remaining asset at the current round; when the answer is yes, we know that there will be a higher price later and should defer trading to at least the next round. It looks helpful at the first sight, and also proven to improve the performance of the secretary problem [2], however, we show that the competitive ratio will not be improved in our case. But when the constant is positive (i.e., $k > 0$), we show that it will strictly improve over the lower bound $1 + \ln \theta$.

Now the question becomes, does the time of querying matter? It turns out that other than asking the query at the first seen price, a deterministic query time does not work, and one must resort to randomized algorithms. We provide a randomized algorithm and show that its competitive ratio breaks the lower bound.

2 RESULTS AND IMPLICATIONS

Below we list the results without proofs due to the page limit.

- We present one deterministic algorithm with the above query model and show that it breaks the lower bound $1 + \ln \theta$ if $k > 0$. We also show that no other deterministic algorithm can use the defined query to improve over $1 + \ln \theta$.
- We propose a randomized algorithm, which draws a random threshold from a uniform distribution and asks the oracle when the current price exceeds the threshold, and show that it will improve over the optimal competitive ratio $1 + \ln \theta$. We also show that if $k = 0$, the performance of the designed algorithm exactly matches the lower bound.
- For online conversion under unknown horizon and known ratio, if we ask in the beginning whether the maximum price exceeds $\sqrt{\theta} p_1$ or not, where p_1 is the first price seen, the competitive ratio of $1 + \ln \theta$ is improved to $1 + \ln \sqrt{\theta}$.

The query-based prediction models were first studied by theoretical computer scientists and viewed as a new complexity measure - query complexity of online problems. After that, specific query models were typically studied with the assumption that it reveals an important global parameter of the instance, as in [1]. Here, we would like to initiate the discussion on the following questions: What if queries only provide local information, say a comparison of the future with the present?

More broadly, what if the oracle is restricted in its power and also subject to a querying budget? How can such query model assist other online problems? What we can be certain about is that a new instance-dependent type of analysis will be needed.

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