Learning-Augmented Algorithms for the Bahncard Problem

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1 Introduction

The Bahncard is a railway pass of the German railway company, which provides a discount on all train tickets for a fixed time period of pass validity. When a travel request arises, a traveler can buy the train ticket with the regular price, or purchase a Bahncard first and get entitled to a discount on all train tickets within its valid time. The Bahncard problem is an *online* cost minimization problem, whose objective is to minimize the overall cost of pass and ticket purchases, without knowledge of future travel requests [3]. It is a generalization of the ski-rental problem and has applications in bandwidth cost minimization, cloud instance reservation, and data migration etc.

Bamas et al. [1] designed a primal-dual-based learning-augmented algorithm assuming given a prediction of the optimal solution, i.e., an optimal collection of times to purchase Bahncards such that the total cost is minimized. Their approach faces three critical issues. First, it is limited to scenarios with slotted time and uniform ticket prices for all travel requests. The discretization is necessary to formulate an integer program and enable the application of the primal-dual method. Second, the consistency and robustness results are weak in that they only hold under the condition that the cost of a Bahncard goes towards infinity, in which case the optimal solution is to never buy any Bahncard. Last, their algorithm demands a complete solution as input advice, which implies a predicted *complete* sequence of travel requests over an arbitrarily long timespan, which is impractical for real employment. Drygala et al. [2] focused on how many predictions are required to gather enough information to output a near-optimal Bahncard purchasing schedule, assuming all predictions are correct. The learning-augmented algorithm proposed takes a suggested sequence of buying times as input and assumes a slotted time setting, thereby sharing similar drawbacks to [1].

In this paper, we adopt a different technical perspective to study the Bahncard problem in the learning-augmented setting. We address the general scenario where travel requests may arise at any time with diverse ticket prices. We develop an algorithm, named PFSUM, which takes short-term predictions on future trips as inputs. We derive its competitive ratio under any prediction error that is naturally defined as the difference between the predicted and true values.

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2 Main Results

Formally, the Bahncard problem is instantiated by three parameters (C, β, T), meaning that a Bahncard costs C, reduces any (regular) ticket price p to βp for some $0 \leq \beta < 1$, and is valid for a time period of T. $\gamma := C/(1-\beta)$ is known as the break-even point, i.e., the threshold to purchase a Bahncard at time t for minimizing the cost incurred in the time interval $[t, t+T)$.

Fleischer [3] was the first to study the Bahncard problem. By extending the optimal 2-competitive break-even algorithm for ski-rental, he proposed an optimal $(2-\beta)$-competitive deterministic algorithm named SUM. SUM purchases a Bahncard at time t when the total regular ticket price actually paid in the past interval $(t-T, t]$ is at least γ.

We assume that at any time t when a travel request arises and there is no a valid Bahncard, a prediction on the total (regular) ticket price of all travel requests in the upcoming interval $[t, t+T)$ is made. With this prediction, an intuitive algorithm design FSUM (Future SUM) is to purchase a Bahncard at time t when the predicted total ticket price in $[t, t+T)$ is at least γ. We show that FSUM is $2/(1+\beta)$-consistent which is generally better than SUM's competitive ratio since $2/(1+\beta) < 2-\beta$ always holds for $0 < \beta < 1$. However, FSUM fails to achieve any bounded robustness because it completely ignores the historical information in the Bahncard purchase condition. Thus, the worst case is that the actual ticket cost in the prediction window is close to 0, while the predictor forecasts that it exceeds γ, in which case hardly anything benefits from the Bahncard purchased.

We note that SUM achieves a decent competitive ratio because a Bahncard is purchased only when the total regular ticket price actually paid in the past interval $(t-T, t]$ is at least γ, so that the Bahncard cost can be charged to such total ticket price in the competitive analysis. Motivated by this observation, we propose a new algorithm PFSUM (Past and Future SUM), in which the Bahncard purchase condition incorporates the ticket costs in both a past time interval and a future prediction window. PFSUM purchases a Bahncard at time t when (i) the total ticket price in the past interval $(t-T, t]$ is at least γ and (ii) the predicted total ticket price in $[t, t+T)$ is also at least γ. Note that the first condition is different from the condition for SUM to purchase a Bahncard: PFSUM considers the total ticket price in the past interval $(t-T, t]$, but SUM considers only the total regular ticket price actually paid in the past interval $(t-T, t]$. The rationale is that the regular ticket price actually paid is not covered by any Bahncard, so requiring it to be at least γ for Bahncard purchase would make the algorithm less effective in saving cost. Compared with FSUM, though PFSUM has an additional condition for purchasing the Bahncard, it is somewhat surprising that PFSUM achieves the same consistency as FSUM.

Denoting by η the maximum prediction error, we derive the competitive ratio of PFSUM as

$$\mathrm{CR}_{\mathrm{PFSUM}}(\eta) = \begin{cases} \frac{2\gamma+(2-\beta)\eta}{(1+\beta)\gamma+\beta\eta} & 0 \leq \eta \leq \gamma, \\ \frac{(3-\beta)\gamma+\eta}{(1+\beta)\gamma+\beta\eta} & \eta > \gamma. \end{cases}$$

The result shows that PFSUM is $2/(1+\beta)$-consistent and $1/\beta$-robust, and its competitive ratio degrades smoothly as the prediction error increases. We conduct extensive experiments to show that PFSUM outperforms existing algorithms.

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