From Prediction to Performance: Consistency and Robustness in Online Knapsack

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1 Introduction and Problem Formulation

The classic online knapsack problem (OKP) seeks to maximize the value of accepted items from an arrival sequence under capacity constraints. We design learning-augmented algorithms using succinct predictions of the offline optimal's critical value $\hat{v}$, achieving near Pareto-optimal consistency-robustness trade-offs (see Table 1). OKP models tasks in online advertising [6], resource management [5], pricing [1], and supply chains [3]. Unlike classical competitive analysis [2, 4, 6], we incorporate predictions to overcome worst-case limits.

Consider a knapsack with capacity 1 (w.l.o.g.). Items arrive online, each with a value v_i and weight w_i. Upon arrival, an algorithm decides $x_i \in \mathcal{X}_i$, the acceptance of item i, without future knowledge. $\mathcal{X}_i$ denotes feasible decisions for item i. Each decision x_i yields profit $x_i v_i$, aiming to maximize total profit under capacity constraints. When $\mathcal{X}_i = \{0, w_i\}$, the algorithm decides to pack or reject the entire item (OIKP). For $\mathcal{X}_i = [0, w_i]$, fractional packing is allowed (OFKP). We use OKP for both problems unless otherwise specified. In the offline case where items are known apriori, an OKP instance $\mathcal{I} = \{(v_i, w_i)\}_{i \in [n]}$ corresponds to:

$$\max_{\{x_i\}_{i\in[n]}} \sum_{i=1}^{n} x_i v_i, \text{ s.t.} \sum_{i=1}^{n} x_i \leq 1, x_i \in \mathcal{X}_i : \forall i \in [n]. \tag{1}$$

The *critical value* is the smallest value among the items (possibly fractionally) packed. Also, assume that all item values lie within $[L, U]$ for all $i \in [n]$. Note that L and U are not related to the predicted interval $[\ell, u]$.

2 Main Results

We introduce algorithms for the online knapsack problem (OKP) under several prediction models.

In the paper, trusted predictions refer to settings where the provided prediction (either a point or interval estimate of the critical value) is assumed to be accurate and can be used directly by the algorithm; this aligns with the algorithms with advice literature, such as [2], where advice is reliable and often strong. In contrast, untrusted predictions correspond to the learning-augmented setting, where predictions may be inaccurate, and the algorithm must achieve a balance between consistency (performance under correct predictions) and robustness (performance under adversarial or incorrect predictions), reflecting the framework of consistency-robustness trade-offs.

PP-b is a simple reserve-then-greedy algorithm for trusted point predictions, while PP-a improves upon it with a reserve-while-greedy strategy that achieves the optimal competitive ratio. For trusted interval predictions, IPA matches the theoretical lower bound by leveraging both endpoints of the

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Table 1. Summary of contributions.

Prediction Model	Algorithm	Upper Bound	Lower Bound
Point Prediction	PP-b , PP-a	$2, 1+\min\{1,\hat{\omega}\}$	$1+\min\{1,\hat{\omega}\}$ (Theorem 2.1)
Interval Prediction	IPA	$2+\ln(u/\ell)$ (Theorem 2.2)	$2+\ln(u/\ell)$ (Theorem 2.3)
Consistency-robustness	MIX	$2/\lambda$-consistent ($\lambda \in (0,1)$) $\frac{\ln(U/L)+1}{1-\lambda}$-robust (Theorem 2.4)*	$2/\lambda$-consistent ($\lambda \in (0,1)$) $\Omega\left(\frac{\ln(U/L)+1}{1-\lambda}\right)$-robust (Theorem 2.5)*

Notes: • Results are for online fractional knapsack (OFKP); integral conversion incurs small loss under standard assumptions.
• $\hat{\omega}$: total weight of items with critical value $\hat{v}$.
• ℓ, u: bounds on predicted critical value $\hat{v}$.
• $\lambda \in (0,1)$: trust hyperparameter in learning-augmented algorithms.
• * indicates value-bounded setting with item values in $[L, U]$.

interval. To handle untrusted predictions, MIX combines a robust baseline with a trusted algorithm, balancing consistency and robustness through a tunable trust parameter. Finally, Fr2Int converts fractional solutions into integral ones when item weights are small, extending the results to the integral setting. Below we mention our most significant results.

Theorem 2.1. *Given an exact prediction (trusted point prediction) on the critical value $\hat{v}$, no online algorithm for* OFKP *can achieve a competitive ratio smaller than $1+\min\{1,\hat{\omega}\}$, while* PP-a *is $1+\min\{1,\hat{\omega}\}$-competitive.*

Theorem 2.2. *Given an interval prediction $[\ell, u]$ and an algorithm A for* OFKP *with a worst-case competitive ratio of α,* IPA *is $(\alpha+1)$-competitive (if we use* ZCL *we get $(2+\ln(u/\ell))$-competitive) .*

Theorem 2.3. *Given a trusted interval prediction $[\ell, u]$, no online algorithm for* OFKP *can achieve a competitive ratio better than $(2+\ln(u/\ell))$.*

Our MIX algorithm combines ZCL, the optimal $(\ln(U/L)+1)$-competitive OFKP algorithm [6], with one of the trusted OFKP prediction algorithms that we devise (e.g., PP-a, IPA, see Table 1).

Theorem 2.4. MIX *is $\frac{\ln(U/L)+1}{(1-\lambda)}$-robust and $\frac{c}{\lambda}$-consistent for* OFKP *for any $\lambda \in (0,1)$, where c is the competitive ratio of the inner prediction* ALG *with an accurate prediction.*

Theorem 2.5. *Given an untrusted prediction of the critical value, any γ-robust algorithm for* OKP *(where $\gamma \in [\ln(U/L)+1, \infty)$) is at least η-consistent for $\eta \geq \max\left\{2 - L/U, \frac{1}{1-\frac{1}{\gamma}\ln(U/L)}\right\}$. Furthermore, in the limit as $U/L \to \infty$, any $2/\lambda$-consistent (for some $\lambda \in (0,1)$) algorithm is at least β-robust, for $\beta = \frac{1+\ln(U/L)}{1-\lambda} - o(1) = \Omega\left(\frac{1+\ln(U/L)}{1-\lambda}\right)$.*

Theorem 2.6. *Given a γ-competitive online algorithm* ALG *for* OFKP *and fixed parameter $\delta > 0$, if the maximum item weight of* OIKP *is upper bounded by $\epsilon < 1/\lceil\log_{(1+\delta)} U/L\rceil+1$, then the algorithm* Fr2Int-ALG *is $\gamma \cdot 1+\delta/1-\epsilon(\lceil\log_{(1+\delta)} U/L\rceil+1)$ competitive for* OIKP.

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